Analytical holographic superconductors in AdS_N topological Lifshitz black holes

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Abstract

We present the analytic Lifshitz solutions for a scalar field model non minimally coupled with the abelian gauge field in N dimensions. We also consider the presence of cosmological constant Λ. The Lifshitz parameter z appearing in the solution plays the role of the Lorentz breaking parameter of the model. We investigate the thermodynamical properties of the solutions and discuss the energy issue. Furthermore, we study the hairy black hole solutions in which the abelian gauge field breaks the symmetry near the horizon. In the holographic picture, it is equivalent to a second order phase transition. Explicitly we show that there exist a critical temperature which is a function of the Lifshitz parameter z. The system below the critical temperature becomes superconductor, but the critical exponent of the model remains the same of the usual holographic superconductors without the higher order gravitational corrections, in agreement with Gainsbourg-Landau theories.

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1 Introduction

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1] has many useful applications in condensed matter physics, especially for studying scale-invariant strongly-coupled systems, for example, and low temperature systems near quantum criticality [2]. There are many attempts to relate the condensed matter problems to their gravitational duals. Since the high-T_c superconductors are shown to be in the strong coupling regime, the BCS theory fails and one expects that the holographic method could give some insights into the pairing mechanism in the high-T_c superconductors. From the (d dimensional) field theory point of view, superconductivity is characterized by condensation of a generally composite charged operator $\hat{O}$ in low temperatures $T < T_c$. In the gravitationally dual ($d + 1$ dimensional) description of the system, the transition to the super conductivity is observed as a classical instability of a black hole in an anti-de Sitter (AdS) space against perturbations by a charged scalar field ψ. The AdS/CFT correspondence relates the quantum dynamics of the boundary operator $\hat{O}$ to a simple classical dynamics of the bulk scalar field ψ [3, 4]. Various holographic superconductors have been studied in Einstein theory [5, 6] or extended versions as Gauss-Bonnet (GB) [7, 8], Weyl corrected ones [9–11], with magnetic field in the bulk action [12–18] and even in the non relativistic model of gravity as, for example, in Horava-Lifshitz theory [19, 20]. In recent years, holographic method have been used to study non-relativistic system [21]. In the framework of condensed mater theory, different system show a dynamical scaling near fixed points:

$$t \to \lambda^z t, \quad x_i \to \lambda x_i, \quad z \neq 1. \tag{1.1}$$

As a consequence, instead obeying the conformal scale invariance $t \to \lambda t, x_i \to \lambda x_i$, the temporal and the spatial coordinates scale anisotropically. The Lifshitz topological black holes and charged Lifshitz black holes have previously been discussed in Refs. [22, 23].

In the present paper we would like to study holographic superconductors in a new background. This set up of the s-wave holographic superconductors uses the Lifshitz AdS black hole as the gravitational bulk metric in the probe limit. In the Section 2, we introduce a scalar field model non minimally coupled with the abelian gauge field in the presence of cosmological constant in N dimensions and we derive static, (pseudo-)spherically symmetric (SSS) solutions with various topologies. In particular, we will be interested in the AdS-black hole (AdS-BH) Lifshitz solutions. In the Section 3, we study the thermodynamical properties of the solutions and obtain the quasi-local generalized Misner-Sharp mass as a Killing conserved charge. We also verify the validity of the Gibbs equation by using the Kodama-Hayward temperature. In Section 4 we study the hairy black hole solutions in which near the horizon the abelian gauge field breaks the symmetry and in Section 5 we explore the scalar condensation in our Lifshitz black hole solutions by analytical approaches. The matching solutions and the critical temperature will be found. Finally, conclusions are given

in the last Section.

2 Bulk asymptotic AdS_N solution

We will consider the N-dimensional action of the following model where the scalar field ϕ is non minimally coupled with electromagnetic potential,

$$I = \int_{\mathcal{M}} d^N x\sqrt{-g}[\frac{(R-2\Lambda)}{2\kappa^2} - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + V(\phi) - \xi \mathrm{e}^{\lambda\phi}(F^{\mu\nu}F_{\mu\nu})] \,. \tag{2.1}$$

Here, g is the determinant of metric tensor, $g_{\mu\nu}$, $\mathcal{M}$ is the space-time manifold, Λ is a cosmological constant, namely $\Lambda = -(N-1)(N-2)/(2L^2)$, L being a length size, and $F_{\mu\nu}$ is the electromagnetic field strength coupled with scalar field ϕ as $\xi\mathrm{Exp}[\lambda\phi](F^{\mu\nu}F_{\mu\nu})$, ξ and λ being generic constants (for example, in the four dimensional Einstein- Maxwell action one has $\xi = 1/4$ and $\lambda = 0$). Here, we use units of $k_B = c = \hbar = 1$ and denote the gravitational constant $\kappa^2 = 8\pi G_N^N \equiv 8\pi(1/M_{Pl}^2)^N$ with the Planck mass of $M_{PL} = G_N^{-1/2} = 1.2\times 10^{19}$GeV. The field is also subjected to a potential $V(\phi)$.

We look for static, (pseudo-)spherically symmetric (SSS) solutions with various topologies, and write the metric element as

$$ds^2 = -e^{2\alpha(r)}B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\sigma^2_{N-2,k}\,, \tag{2.2}$$

where $\alpha(r)$ and $B(r)$ are function of r only and $d\sigma^2_{N-2,k}$ represents the metric of a topological $(N-2)$-dimensional surface parametrized by $k = 0, \pm 1$, such that the manifold will be either a sphere S_{N-2} (for $k = 1$), a torus T_{N-2} (for $k = 0$) or a compact hyperbolic manifold Y_{N-2} (for $k = -1$). In particular, we will be interested in the Lifshitz solutions, where $\alpha(r) = \log r^{z/2}$, being z the redshift parameter. In this case, for power counting renormalizability in N dimension, if we assume that the interaction potential can be expanded as $V(\phi) = \Sigma_{m=0}^K g_m\phi^m$, where ϕ^m are the polynomial terms of the series and g_m suitable coefficients, by the dimensional engineering we get [24]

$$[g_m] = [m]^{\frac{[N+z-1-m(N-1-z)/2}{z}}\,.$$

Such kind of theory is renormalizable if the couplings have non-negative momentum (here mass) dimension, so that we have two possibilities, namely

$$K = \frac{2(N-1+z)}{N-1-z}, \quad z < N-1\,,$$
$$K = \infty, \quad z \geq N-1.$$

The above constraints are valid for any scalar theory under the Lifshitz scaling of the coordinates.

We propose the potential in the following form,

$$V(\phi) = V_0 \mathrm{e}^{\gamma\phi}\,, \tag{2.3}$$

where V_0 and γ are generic parameters. Now, by comparing the exponential potential (2.3) with the polynomial form, we observe that in fact $K = \infty$, so that the theory (2.1) is renormalizable for $z \geq N - 1$.

With the metric ansatz (2.2), the scalar curvature reads

$$R = -3B'\alpha' - 2\,B\alpha'^2 - B'' - 2\,B\alpha'' - (N-2)\left[\frac{2}{r}B' + 2\,\frac{B\alpha'}{r} - \frac{(N-3)}{r^2}(k-B)\right]\,, \tag{2.4}$$

where the prime index denotes the derivative with respect to r. Where is not necessary, the argument of the functions $\alpha(r)$ and $B(r)$ will be dropped.

Due to the $SO(N-2)$ symmetry, it is easy to see that the only non vanishing components of the electromagnetic field in N-dimension are

$$F_{01} = \frac{e^{\alpha(r)}Q}{r^{N-2}}\,, \quad F^{01} = -\frac{e^{-\alpha(r)}Q}{r^{N-2}}\,, \tag{2.5}$$

Q being the electric charge of electromagnetic field multiplied to some suitable dimensional positive paramter. By assuming $\phi = \phi(r)$ and by plugging the above expressions into the action (2.1), making a partial integration, we get the following effective Lagrangian,

$$\mathcal{L}_{\text{eff}} = \mathrm{e}^{\alpha}r^{N-2}\left[\frac{(N-2)(N-3)(k-B)}{r^2} - \frac{(N-2)B'}{r} - 2\Lambda - \frac{B\tilde{\phi}'^2}{2} + \tilde{V}_0\mathrm{e}^{\gamma\phi} + 2\xi\frac{\mathrm{e}^{\lambda\phi}\tilde{Q}^2}{r^{2(N-2)}}\right]\,, \tag{2.6}$$

where, for simplicity, we have putted $\tilde{\phi} = 2\kappa^2\phi$, $\tilde{V}_0 = 2\kappa^2 V_0$ and $\tilde{Q}^2 = 2\kappa^2 Q$. Making the variation with respect to $\alpha(r)$ and $B(r)$ we get the Equations of motion (EOMs),

$$B'r - (N-3)(k-B) - \frac{r^2}{(N-2)}\left[-2\Lambda - \frac{\tilde{\phi}'^2 B}{2} + \tilde{V}_0\mathrm{e}^{\gamma\phi} + \frac{2\xi\mathrm{e}^{\lambda\phi}\tilde{Q}^2}{r^{2(N-2)}}\right] = 0\,, \tag{2.7}$$

$$\alpha' - \frac{\tilde{\phi}'^2 r}{2(N-2)} = 0\,. \tag{2.8}$$

This equations correspond to the (t,t) and (r,r) independent components of the usual tensorial form of field equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa^2(T^{\phi}_{\mu\nu} + \xi\mathrm{e}^{\lambda\phi}(4F_{\nu\sigma}\partial_\mu A^\sigma - g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}))\,, \tag{2.9}$$

where $R_{\mu\nu}$ is the N-dimensional Ricci tensor, $T^{\phi}_{\mu\nu}$ is the stress energy tensor of the field ϕ and A_μ

represents the electromagnetic potential such that $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$.

Finally, the equation for $\phi(r)$ reads

$$\tilde{\phi}'' B + \left[\alpha' B + B' + \frac{N-2}{r} B\right] \tilde{\phi}' + \left[\gamma \tilde{V}_0 e^{\gamma\phi} + \frac{2\xi\lambda e^{\lambda\phi}\tilde{Q}^2}{r^{2(N-2)}}\right] = 0 . \tag{2.10}$$

As we stated above, we are interested in the following solutions,

$$\alpha(r) = \log[(r/r_0)^{z/2}] , \tag{2.11}$$

which correspond to the important class of Lifshitz solutions parameterized by red shift z parameter. Here, r_0 is a dimensional constant. From Eq. (2.8) we get

$$\tilde{\phi}(r) = \sqrt{z(N-2)} \log[r/r_0] , \tag{2.12}$$

so that for renormalizable theory ($z > N-1$) we deal with real fields, since in general we take $N > 3$. By using this result, we can solve Eq. (2.7) as

$$\begin{aligned} B(r) &= \frac{2k(N-3)}{z+2N-6} + C\, r^{3-N+z/2} - \frac{2r^A \tilde{V}_0}{(N-2)(6-2A-2N-z)} \\ &\quad - \frac{4\xi\tilde{Q}^2 r^B}{(N-2)(6-2B-2N-z)} - \frac{4\Lambda r^2}{(N-2)(2N-2+z)} . \end{aligned} \tag{2.13}$$

In this equation, C is a free integration constant of the solution and

$$\begin{aligned} A &= \gamma\sqrt{z(N-2)} + 2 , \\ B &= \lambda\sqrt{z(N-2)} + 6 - 2N . \end{aligned} \tag{2.14}$$

For simplicity, in the above expression we have also redefined $r_0^{-\gamma\sqrt{z(N-2)}}\tilde{V} \to \tilde{V}$ and $r_0^{-\lambda\sqrt{z(N-2)}}\tilde{Q}^2 \to \tilde{Q}^2$. Note that if we turn off the electromagnetic and field potentials and then take $\phi = 0$, we recover the Reissner Norstrom solution with cosmological constant,

$$\begin{aligned} ds^2 &= -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\sigma^2_{(N-2),k} , \\ B(r) &= k + Cr^{3-N} - \frac{2\Lambda r^2}{2-3N+N^2} + \frac{2\xi\tilde{Q}^2}{(N-5)(N-2)r^2} . \end{aligned} \tag{2.15}$$

On the other hand, in the presence of the scalar field, the solution (2.13) is acceptable only if the Klein Gordon equation (2.10) for ϕ is also satisfied. In this case, the form of the metric is given by

$$ds^2 = -\left(\frac{r}{r_0}\right)^z B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\sigma^2_{N-2,k} . \tag{2.16}$$

Let us see for two different cases.

- Absence of cosmological constant. If $\Lambda = 0$, Eq. (2.10) is satisfied by choosing

$$\gamma = -\frac{2}{\sqrt{z(N-2)}}\,,$$
$$\lambda = \frac{(2N-6)}{\sqrt{z(N-2)}}$$
$$\tilde{V}_0 = \frac{k(N-3)(N-2)z + 2\tilde{Q}^2(2N-6+z)\xi}{(2-z)}\,. \tag{2.17}$$

  The solution becomes

$$B(r) = -\frac{4[k(N-3)+2\tilde{Q}^2\xi]}{(z-2)(2N-6+z)} + Cr^{3-N-z/2}\,. \tag{2.18}$$

- Cosmological constant $\Lambda \neq 0$. One simple solution is given by

$$\lambda = \frac{2N-4}{\sqrt{z(N-2)}}\,,$$
$$\gamma = -\frac{2}{\sqrt{z(N-2)}}\,,$$
$$z = \frac{2\,(N-2)\,\tilde{Q}^2\xi}{\Lambda - \tilde{Q}^2\xi}\,,$$
$$\tilde{V}_0 = -\frac{k(N-3)(N-2)^2\tilde{Q}^2\xi}{(N-1)\tilde{Q}^2\xi - \Lambda}\,. \tag{2.19}$$

  The solution reads

$$\begin{aligned} B(r) \;&=\; \frac{1}{(2N-6+z)}\left[2k(N-3) + \frac{2\tilde{V}_0}{(N-2)}\right] + Cr^{3-N-z/2} \\ &\quad -\frac{4r^2(\Lambda - \tilde{Q}^2\xi)}{(N-2)(2N-2+z)}\,, \end{aligned} \tag{2.20}$$

  which asymptotically is a de Sitter/Anti de Sitter (dS/AdS) solution. Note that if $\Lambda = 0$, we get $z = -2(N-2)$, and, for $N > 2$, the solution is not accettable, being ϕ in Eq. (2.12) immaginary, and the theory becomes non renormalizable. In principle, for any dimension N and for any choice of z we can obtain the corresponding Lifshitz solution by setting the values of γ, λ, Q and V_0 appearing in the field lagrangian.

  The solution (2.20) can be asymptotically AdS and will furnish our bachground in studying holographic superconductors.

3 Black hole solutions and thermodynamics

The solutions derived in the previous Section may describe charged black holes (BH) in N-dimensional manifolds in the presence of scalar field non minimally coupled with electrodynamic potential. We recall that event horizon exists as soon as there exists a positive solution r_+ of

$$B(r_+) = 0\,, \qquad B'(r_+) \gneq 0\,. \tag{3.1}$$

We require $B(r_+) \neq$ to avoid the non extremal BHs.

In the case of solution (2.18) one has

$$r_+ = \left[\frac{4k(N-3) + 8\tilde{Q}^2\xi}{C(z-2)(2N-6+z)}\right]^{\frac{1}{3-N-z/2}}\,, \tag{3.2}$$

and since $z > 0$, we must require $k/C > 0$ when $N > 3$. However, in order to have $B'(r_+) > 0$, we see that only in the topological case $k = -1$ we obtain a BH solution.

Concerning the solutions (2.15) and (2.20), it is always possible to describe topological BHs by making an appropriate choice of the parameters. For example, in the case of solution (2.20) with

$$\frac{1}{(2N-6+z)}\left(2k(N-3) + \frac{2\tilde{V}_0}{(N-2)}\right) > 0\,, \quad \frac{4r^2(\Lambda - \tilde{Q}^2\xi)}{(N-2)(2N-2+z)} < 0\,,$$

the equation $B(r) = 0$ has two roots, namely $r_\pm$, the first one corresponding to the event horizon of the black hole ($B'(r_+) > 0$) and the second one to the (Anti-de Sitter) horizon of the cosmological background where the black hole is immersed ($B'(r_-) < 0$).

In the following, we will assume to deal with solutions whose parameters satisfy the conditions (3.1) for some value of $r = r_+$.

Let us study some physical propriety of these black holes. Since the field equations (2.9) of the theory are second order differential equations, we can easily derive a conserved current whose charge may be identified with the mass of the BHs. We will follow the approach proposed by Wheeler for Lovelock theories [25] in Ref. [26]. For simplicity, we denote with $\mathcal{G}_{\mu\nu}$ the Einstein tensor plus the contribute of cosmological constant, namely

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\,. \tag{3.3}$$

By means of time-like Killing vector field $K^\nu = (1, \vec{0})$ in N-dimension, in the case of static metric (2.2), one can construct the conserved current

$$J_\mu := \mathcal{G}_{\mu\nu}K^\nu\,, \tag{3.4}$$

such that

$$\nabla_\nu J^\nu = 0\,, \tag{3.5}$$

being $\mathcal{G}_{\mu\nu}$ a conserved quantity. A direct evaluation of J_0 via (3.3) leads to

$$J_0 := G_{00}K^0 = (\mathrm{e}^{2\alpha(r)}B(r))\frac{(N-2)}{2r^{N-2}}\frac{d[r^{N-1}W(r)]}{dr}\,, \tag{3.6}$$

where

$$W(r) = [k - B(r)]r^{-2} - \frac{2\Lambda}{(N-2)(N-1)}\,. \tag{3.7}$$

The current J_μ gives rise to a Killing conserved charge. This corresponds to the quasi-local generalized Misner-Sharp [27] mass which reads

$$E_{MS}(r) \equiv -\frac{1}{\kappa^2}\int_\Sigma J^\mu d\Sigma_\mu = \frac{(N-2)V_{N-2,k}}{2\kappa^2}\int_0^r d\rho\frac{d(\rho^{N-1}W)}{d\rho} = \frac{(N-2)V_{N-2,k}}{2\kappa^2}r^{N-1}W(r)\,, \tag{3.8}$$

where Σ is a spatial volume at fixed time, $d\Sigma_\mu = (d\Sigma, \vec{0})$, and $V_{N-2,k}$ is the $N-2$ dimensional volume depending on the topology. For example, in the case of the sphere with $k = 1$, one has $V_{N-2,1} = 2\pi^{(N-1)/2}/\Gamma((N-1)/2)$, with $\Gamma(z)$ the Euler-Gamma function.

In particular, on shell, that is at the horizon $r = r_+$ such that $B(r_+) = 0$, the quasi local energy is identified with the black hole energy E which reads

$$E := E_{MS}(r_+) = \frac{(N-2)V_{N-2,k}}{2\kappa^2}\left(kr_+^{N-3} - \frac{2\Lambda}{(N-2)(N-1)}r_+^{N-1}\right)\,. \tag{3.9}$$

For example, in the vacuum case of Eq. (2.15) with $\tilde{Q} = 0$ one has

$$r^{N-1}W(r) = -C\,, \tag{3.10}$$

and the black hole energy reads

$$E = -\frac{(N-2)V_{N-2,k}}{2\kappa^2}C\,, \tag{3.11}$$

so that the constant of integration is related with the mass of the BH.

We note that expression (3.9) correctly returns the Misner-Sharp mass for asymptotically flat solutions ($\Lambda = 0$) in vacuum or in the presence of matter. In particular, for $N = 4$ and $k = 1$, by explicitly writing the Newton Constant, we get the familiar result $E = r_+/(2G_\mathrm{N})$, which corresponds, in the vacuum case, to $E = -C/(2G_\mathrm{N})$.

Now, let us show that the Gibbs equation $TdS = dE - pdV$ holds true for the black holes

described by the model (2.1), with the Killing energy E obtained below, and the pressure p given by electromagnetic and scalar fields. T and S are the temperature and the entropy of the black hole, and V is the volume enclosed by the horizon in $N-1$ dimensional space.

For Lovelock gravity the validity of the First Law of black hole thermodynamics has been investigated in several places [28–30]. For our static non vacuum case we present a simple derivation from the first EOM (2.7) evaluated on the horizon $r=r_+$,

$$\begin{aligned}&\frac{(N-2)V_{N-2,k}B'(r_+)r_+^{N-3}}{2\kappa^2}-\frac{(N-2)V_{N-2,k}}{2\kappa^2}\left(kr_+^{N-3}-\frac{2\Lambda}{(N-2)(N-1)}r_+^{N-1}\right)\\&-\frac{V_{N-2,k}r^{N-2}}{2\kappa^2}(\tilde{V}_0\mathrm{e}^{\gamma\phi}+\frac{2\xi\mathrm{e}^{\lambda\phi}\tilde{Q}^2}{r_+^4})=0\,.\end{aligned}\tag{3.12}$$

Here, we have used the fact that $B(r_+)=0$.

All thermodynamical quantities associated with black holes solutions can be computed by standard methods. The entropy can be calculated by the Wald method [31–33] and reads

$$S_W=\frac{2\pi V_{N-2,k}}{\kappa^2}r_+^{N-2}\,.\tag{3.13}$$

Furthermore, for the static metric (2.2) it is possible to find a characteristic temperature related to the event horizon. A natural choice is to take the so called Killing/Hawking temperature [34]

$$T_K:=\frac{\kappa_K}{2\pi}=\frac{\mathrm{e}^{\alpha(r_+)}}{4\pi}B'(r_+)\,,\tag{3.14}$$

whose validity may be justified making use of derivations of Hawking radiation [35] or by eliminating the conical singularity in the corresponding Euclidean metric [36] or making use of the tunneling method [37,38]. In the above expression, κ_K denotes the Killing surface gravity, namely $\kappa_K=\mathrm{e}^{\alpha(r_+)}B'(r_+)/2$, derived from the relation $K^\mu\nabla_\mu K^\nu=\kappa_K K^\nu$, where $K^\nu=(1,\vec{0})$ is the time-like Killing vector field.

However, we would like to remind that in the spherical symmetric, dynamical case, the real geometric object which generalizes the Killing vector field is the Kodama field with a related conserved current and a related Kodama surface gravity κ_H [39]. In such a case, a natural definition of the temperature for dynamical black holes reads as $T_H=\kappa_H/2\pi$, where T_H is the Kodama-Hayward temperature, in analogy with the static case.

For spherically symmetric statical metric, the Kodama vector differs from the Killing one as $\mathcal{K}^\nu=\mathrm{e}^{-\alpha(r)}K^\nu$ and as a consequence one finds

$$T_H:=\frac{\kappa_H}{2\pi}=\frac{1}{4\pi}B'(r_+)\,,\tag{3.15}$$

namely $T_H=\mathrm{e}^{-\alpha(r)}T_K$ and in principle definitions (3.14) and (3.15) are different. In vacuum case

where $\alpha(r) = 0$, the two temperatures coincide, but for "dirty" BHs (i.e., in the presence of matter) as the ones we are considering, $T_K \neq T_H$. This is related with the fact that the Killing vector cannot be defined unambiguously when the space-time is not asymptotically flat. We stress that all derivations of Hawking radiation lead to a semi-classical expression for the black hole radiation rate Γ,

$$\Gamma \equiv \mathrm{e}^{-\frac{\Delta E_K}{T_K}}\,, \tag{3.16}$$

in terms of the change ΔE_K of the Killing energy E_K, but if one uses the Kodama energy E_H for the emitted particle, one has

$$\Gamma \equiv \mathrm{e}^{-\frac{\Delta E_H}{T_H}}\,. \tag{3.17}$$

This fact derives by the relationship $\Delta E_H = \mathrm{e}^{-\alpha(r)} \Delta E_K$. From the Eqs. (3.16)-(3.17), one arrives at the identity

$$\frac{\Delta E_H}{T_H} = \frac{\Delta E_K}{T_K}\,, \tag{3.18}$$

so that the tunneling probability is invariant under different choices of the temperature.

In Ref. [40] an attempt to identify the mass of static BHs in modified theories of gravity as the integration constant which appears in the vacuum solutions has been done. This result has been derived by the EOMs and seems in favor of the Killing temperature with respect to the Kodama-Hayward one, but here, for our non vacuum solutions, Eq. (3.12) suggests the use of the Kodama temperature (3.15), in the attempt to recover the Gibbs relation. In fact, by making use of the BH entropy (3.13), one can rewrite Eq. (3.12) as

$$T_H dS_W = dE + pdV\,, \tag{3.19}$$

where E is the BH energy (3.9), V is the volume enclosed by the horizon in $N-1$-dimensional space, $V = V_{N-2,k} r^{N-1}/(N-1)$, and $p = p_\phi + p_{EM}$ is the working term given by scalar (p_ϕ) and electromagnetic (p_{EM}) field pressures,

$$\begin{aligned} p_\phi &= -V(\phi) = -V_0 \mathrm{e}^{\gamma\phi} \\ p_{EM} &= \xi F^{\mu\nu} F_{mu\nu} = -\frac{2\xi Q^2}{r^4}\,. \end{aligned} \tag{3.20}$$

Here, we have reintroduced $V_0 = \tilde{V}_0/(2\kappa^2)$ and $Q^2 = \tilde{Q}^2/(2\kappa^2)$. In this case, the Gibbs equation holds true.

In conclusion, we prefer to use the Hayward temperature T_H (3.15) in analogy with therodynamic, by starting from the robust defininitions of the neergy as the charge of a conserved current and the entropy via Wald method.

4 s-wave Holographic superconductors in probe limit

Our goal in the following Sections is to apply the black hole solution with the Lifshitz scaling which it has been obtained before to study the holographic picture of a high temperature superconductor via gauge/gravity duality. In brief, by holographic superconductor we mean a condensed matter system under second order phase transition whose physical properties can be described by studying the dynamics of gauge field on the black hole background in the bulk. It is proved that by direct applying the AdS/CFT conjecture one can interpret the asymptotic behavior of the gauge fields on the AdS boundary as the expectation values of some physical operators. The expectation values of such scalar operators are dual to the super current in the s-wave high temperature type II superconductors. Here s-wave refers to the $U(1)$ gauge field. The gauge field can be $SU(2)$ and corresponds to the Yang-Mills fields. Such holographic models are called p-wave, because super current is a vector and the condensation happens usually for one homogenous component of it. To apply the gravitational model to the superconductors, we will modify our gravitational model (2.1) by adding a new matter field subjected to some abelian gauge field. By starting from the backgound metric of gravitational action previoulsy studied, we will investigate the scalar condensation of the new field and we will show that some phase transition occurs.

We easily see that Eq. (2.20) may describe an (dS/AdS) black hole solution for our non minimally coupled gravity model in N dimensional bulk. We are interested in Anti de Sitter solutions. If we want to relate our gravitational system with a strongly correlated system in the dual quantum theory, we need to describe the dual quantum operators via CFT. The condensed matter system dual to our AdS solution can be addressed by holographic superconductors. In what follows, we will study the formation of the hairy BHs. The phenomenon is given by a second order phase transition and can be described by the holographic methods of the AdS/CFT.

At first, we note that the solution (2.20) with $\Lambda \neq 0$ may be asymptotically topological AdS and it can be considered as the gravitational part of the holographic superconductor in the bulk. In fact, we take the gravity bulk as the charged topological black hole with a non zero, negative effective cosmological constant

$$\Lambda_{eff} = -\frac{4(\Lambda - \tilde{Q}^2\xi)}{(N-2)(2N-2+z)} < 0\,. \tag{4.1}$$

It is very interesting that the $U(1)$ reduced charge $\tilde{Q}$ is combined with cosmological constant in the solution, producing the AdS background. It is useful to rewrite the solution (2.20) as

$$B(r) = \frac{1}{(2N-6+z)}(2k(N-3) + \frac{2\tilde{V}_0}{(N-2)}) + Cr^{3-N-z/2} + \frac{r^2}{l_{eff}^2}\,, \tag{4.2}$$

where

$$l_{eff} = \frac{1}{2}\sqrt{\frac{(N-2)(2N-2+z)}{\tilde{Q}^2\zeta - \Lambda}} \tag{4.3}$$

is the effective lenght scale. We note that for $N = 4$ and $z = 0$, solution (4.3) reduces to the usual Schwarzschild-AdS form with $l_{eff} = \sqrt{3/\Lambda}$. In fact, when we turn off the electromagnetic field, the form of the effective length scale is the same as the one of an AdS uncharged black hole.

We want to find the second order phase transition in the bulk theory by studying the boundary operators. In particular, we want to study the role of the Lifshitz scaling z related to the critical temperature T_c and the condensation of the dual operators $< \mathcal{O}_\pm >$ (see also Refs. [41–43]). As a starting point, in order to discuss the superconducting phase via holographic picture, we need a scalar field $\psi(r)$, with mass upper than the Breitenlohner-Freedman (BF) bound [45], and an abelian gauge field $\mathcal{A}_\mu$, which is non minimally coupled with the scalar field, so that we modified the (2.1) by introducing a new matter Lagrangian in the following form [46]

$$\mathcal{L}_m = -\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - |D_\mu\psi|^2 - m^2\psi^2\,, \tag{4.4}$$

where D_μ is the covariant derivative, $D_\mu = \partial_\mu - iq\mathcal{A}_\mu$, and $\mathcal{F}_{\mu\nu}$ is the electromagnetic field strength related to the abelian field. As a consequence, the total action results to be

$$I_{total} = -\frac{1}{q^2}\int d^N x\sqrt{-g}\mathcal{L}_m + \int_{\mathcal{M}} d^N x\sqrt{-g}\left[\frac{(R-2\Lambda)}{2\kappa^2} - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + V(\phi) - \xi e^{\lambda\phi}(F^{\mu\nu}F_{\mu\nu})\right]. \tag{4.5}$$

In the above expressions, q plays the role of bulk electric charge. We stress that it is not the physical "electric" charge, but it is the dyonic magnetic one. The matter action (4.4) is different from the bulk action. In fact, we add here the non minimally coupled Maxwell field in order to break the $U(1)$ symmetry of the abelian field near the BH horizon. Thus, we will work in the so called probe limit $q \to \infty$, ignoring the back reaction, in the normal phase $\psi = 0$. In this case, the gravity sector decouples from the abelian one and the background metric can be derived from Eq. (2.16) and Eq (2.20).

In the probe limit the EOMs for $\psi(r)$ and $\mathcal{A}_\mu = \phi(r)\delta_{\mu t}$, $\delta_{\mu\nu}$ being the Kroenecker delta function and $\phi(r)$ a general function of r, read in the following forms

$$D_\mu D^\mu\psi - m^2\psi = 0, \tag{4.6}$$

$$\nabla^\mu\mathcal{F}_{\mu\nu} = iq[\psi^* D_\nu\psi - \psi D^*_\nu\psi^*]\,. \tag{4.7}$$

Here, $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$ and we take $\psi^* = \psi$, motivated by the fact that we are free to choose

our gauge. The above equations read in terms of the background metric as

$$\phi'' + \Big(\frac{N-2-\frac{z}{2}}{r}\Big)\phi' - \frac{2q^2\psi^2}{B}\phi = 0\,, \tag{4.8}$$

$$\psi'' + \Big(\frac{N-2+\frac{z}{2}}{r} + \frac{B'}{B}\Big)\psi' + \Big(-\frac{m^2}{B} + \frac{q^2 r^{-z}\phi^2}{B^2}\Big)\psi = 0. \tag{4.9}$$

For $N=4$ and $z=0$ the model is perfectly described by numerical methods [46]. We are interesting in the cases of $z \neq 0$ and $N > 3$.

We need to introduce in Eqs. (4.8)-(4.9) the horizon radius r_+. We rewrite the solution in the following form,

$$B(r) = \tilde{k}\left[1 - \Big(\frac{r}{r_+}\Big)^{3-N-\frac{z}{2}}\right] + \Big(\frac{r_+}{l_{eff}}\Big)^2\left[\Big(\frac{r}{r_+}\Big)^2 - \Big(\frac{r}{r_+}\Big)^{3-N-\frac{z}{2}}\right], \tag{4.10}$$

where

$$\tilde{k} = \frac{1}{2N-6+z}\Big(2k(N-3) + \frac{2\tilde{V}_0}{N-2}\Big). \tag{4.11}$$

It is more convenient to work in terms of the dimensionless parameter $y(r) = r_+/r$, such that $y(r_+) = 1$ and at the infinity $y(r \to +\infty) \to 0^+$.[5] In this case, the equations of motion (4.8) and (4.9) can be expressed as:

$$\phi'' - \Big(\frac{\frac{z}{2}+N-4}{y}\Big)\phi' - \frac{2r_+^2\psi^2}{y^4 B}\phi = 0\,, \tag{4.12}$$

$$\psi'' - \Big(\frac{\frac{-z}{2}+N-4}{y} - \frac{B'}{B}\Big)\psi' - \frac{r_+^2}{y^4}\Big(\frac{m^2}{B} - \frac{y^z\phi^2}{r_+^z B^2}\Big)\psi = 0\,, \tag{4.13}$$

and the metric function reads

$$B(y) = \tilde{k}\Big(1 - y^{N-3+\frac{z}{2}}\Big) + \Big(\frac{r_+}{l_{eff}}\Big)^2\Big(y^{-2} - y^{N-3+\frac{z}{2}}\Big). \tag{4.14}$$

Now, the prime denotes the derivative with respect to $y = y(r)$. We have obtained the basic set up for the holographic superconductors.

[5] In the literature usually the symbol is z, but here we kept z for Lifshitz scaling and we introduced y as the dimensionless radial coordinate.

5 Critical temperature and condensation values by matching method

According to the AdS/CFT correspondence, we can interpret $\langle \mathcal{O} \rangle$, where $\mathcal{O}$ is the operator dual to the scalar field, as the expectation value of the dual current operator on the AdS boundary $r \to +\infty$. Thus, we are going to calculate the condensate $\langle \mathcal{O} \rangle$ for fixed charge density.

Regularity at the horizon, namely $y = 1$, requires

$$\phi(1) = 0\,, \qquad \psi'(1)B'(1) = r_+^2 m^2 \psi(1)\,. \tag{5.1}$$

We want to find approximate solutions around the horizon and asymptotically AdS limit, $y = 1$ and $y = 0$, using Taylor's expansion, then we want to connect these solutions in an arbitrary matching point y_m between $y = 1$ and $y = 0$. In principle, we may do the computations by numerical methods using shooting algorithm, but in this paper we would like to keep the level of the analytical approach. At first, we calculate $B'(1)$ $(= (dB(r_+)/dy)$ directly from Eq. (4.14),

$$B'(1) \;=\; -\tilde{k}(N - 3 + \frac{z}{2}) + \Big(\frac{r_+}{l_{eff}}\Big)^2 \Big(-N + 1 - \frac{z}{2}\Big)\,. \tag{5.2}$$

It is easy to see that Eq. (5.2) is related to the Kodama temperature (3.15) by $B'(1) = -4\pi r_+ T_H$. We will use (5.2) to construct the series solutions for our topological holographic superconductor in the probe limit.

5.1 Solution near the BH horizon

We expand ϕ and ψ near the black hole horizon at $y = 1$ as

$$\phi(y) \;=\; \phi(1) - \phi'(1)(1 - y) + \frac{1}{2}\phi''(1)(1 - y)^2 + \cdots\,, \tag{5.3}$$

$$\psi(y) \;=\; \psi(1) - \psi'(1)(1 - y) + \frac{1}{2}\psi''(1)(1 - y)^2 + \cdots\,. \tag{5.4}$$

From the boundary condition, we know that $\phi(1) = 0$ and for simplicity we put $a := -\phi'(1) < 0$ and $b := \psi(1) > 0$ for the positivity of $\phi(y)$ and $\psi(y)$.

In order to discuss phase transition near the critical points, we need just to keep the second order terms in those series.

First, we compute the 2^{nd} order coefficient of ϕ by using Eq. (4.12) ,

$$\phi''(1) \;=\; -\phi'(1)\Big[-(\frac{z}{2} + N - 4) + \frac{2r_+^2 b^2}{B'(1)}\Big]. \tag{5.5}$$

In this case Eq. (5.3) reads

$$\phi(y) = a\Big[(1-y) + \frac{1}{2}\Big[-(\frac{z}{2}+N-4) + \frac{2r_+^2 b^2}{B'(1)}\Big](1-y)^2\Big]. \tag{5.6}$$

We can calculate the 2nd derivative of ψ from (4.13) in the same way,

$$\psi''(1) \;=\; -\frac{ba^2(r_+)^{2-z}}{(B'(1))^2}\,, \tag{5.7}$$

and write the following series solution for scalar field ψ (5.4)

$$\psi(y) \;=\; b\Big[1 - \frac{m^2 r_+^2}{B'(1)}(1-y) - \frac{a^2 r_+^{2-z}}{2(B'(1))^2}(1-y)^2\Big]\,. \tag{5.8}$$

In the above equations, $B'(1)$ is given by Eq. (5.2).

5.2 Solution near the asymptotic AdS region

The asymptotic regime of the metric function $B(y)$ in the AdS boundary is completely independent on the dimension of the spacetime N. It is trivial to recover the following asymptotic behavior of the metric

$$B(y) \sim y^{-2}\,. \tag{5.9}$$

Here we take $z > 0$ and $N > 3$ to avoid the problems of convergence in AdS/CFT. Consider now the weak field behavior ($r \to +\infty$) of ϕ which depends on the value of dimension N. It is easy to show that this behaviour completely changes at the critical dimension $N_c = 3 - z/2$, namely

$$\phi(y) \;=\; \frac{c_1}{\frac{z}{2}+N-3} y^{\frac{z}{2}+N-3} + c_2, \quad N \neq N_c, \tag{5.10}$$

$$\phi(y) \;=\; c_1 \log(y) + c_2, \quad N = N_c. \tag{5.11}$$

We are interested in the fields with fall off behaviors near $y = 0$, so that we take $N \neq N_c$ case as the physically acceptable solution. It is useful to write (5.10) in terms of the radial coordinate r,

$$\phi(r) \;=\; \frac{c_1 r_+^{\frac{z}{2}+N-3}}{(\frac{z}{2}+N-3)r^{\frac{z}{2}+N-3}} + c_2. \tag{5.12}$$

By writing this solution in terms of the dual physical quantities chemical potential μ and charge density ρ, we obtain

$$\phi(y) \;=\; \mu - \frac{\rho}{r_+^{\frac{z}{2}+N-3}} y^{\frac{z}{2}+N-3}\,, \tag{5.13}$$

where

$$\rho = -\frac{c_1 r_+^{\frac{z}{2}+N-3}}{(\frac{z}{2}+N-3)}\,, \quad \mu = \frac{\rho}{r_+^{\frac{z}{2}+N-3}}\,. \tag{5.14}$$

The second condition derived from the fact that $\phi(r_+) = 0$. Also for the scalar field, near the AdS boundary we can write

$$\psi(y) \;\; = \;\; \frac{<\mathcal{O}_+>}{r^{\Delta_+}} + \frac{<\mathcal{O}_->}{r^{\Delta_-}}\,, \tag{5.15}$$

where $<\mathcal{O}_+>$ and $<\mathcal{O}_->$ are the operators on the boundary. Furthermore, to have a normalizable solution, we will set the source $<\mathcal{O}_->= 0$. By rewriting it in terms of y with the fall off behavior just for operator $\mathcal{O}_+$ we get

$$\psi(y) \;\; = \;\; \frac{<\mathcal{O}_+>}{r_+^{\Delta_+}} y^{\Delta_\pm}. \tag{5.16}$$

Here

$$\Delta_\pm = \frac{1}{2}[(N-1) \pm \sqrt{(N-1)^2 + 4m^2 l_{eff}^2}]\,. \tag{5.17}$$

5.3 Matching and phase transition

In this Section, we will connect the solutions (5.6) and (5.8) with (5.13) and (5.16) at some matching point $y = y_m$. In order to connect those solutions smoothly, we require the following four conditions:

$$\mu - \frac{\rho}{r_+^{z/2+N-3}} y_m^{z/2+N-3} = a\Big[(1-y_m) + \frac{1}{2}\Big[-(\frac{z}{2}+N-4) + \frac{2r_+^2 b^2}{B'(1)}\Big](1-y_m)^2\Big]\,, \tag{5.18}$$

$$-\frac{\rho(z/2+N-3)}{r_+^{z/2+N-3}} y_m^{z/2+N-4} = a\Big[-1-\Big[-(\frac{z}{2}+N-4) + \frac{2r_+^2 b^2}{B'(1)}\Big](1-y_m)\Big], \tag{5.19}$$

$$\frac{<\mathcal{O}_+>}{r_+^{\Delta_+}} y_m^{\Delta_+} = b\Big[1 - \frac{m^2 r_+^2}{B'(1)}(1-y_m) - \frac{a^2 r_+^{2-z}}{2(B'(1))^2}(1-y_m)^2\Big]\,, \tag{5.20}$$

$$\frac{\Delta_+ <\mathcal{O}_+>}{r_+^{\Delta_+}} y_m^{\Delta_+-1} = b\Big[\frac{m^2 r_+^2}{B'(1)} + \frac{a^2 r_+^{2-z}}{(B'(1))^2}(1-y_m)\Big]. \tag{5.21}$$

By combining Eqs. (5.18)-(5.19) we can eliminate ab^2 and one has

$$\mu = \frac{2\,\rho\left((\frac{z}{2}+N-3){y_m}^{\frac{z}{2}+N-3} - (-5+\frac{z}{2}+N){y_m}^{\frac{z}{2}+N-2}\right){r_+}^{-\frac{z}{2}-N+3} + 2\,(1-y_m)y_m a}{4y_m}, \tag{5.22}$$

$$b = \frac{\sqrt{2}}{2r_+}\sqrt{\frac{\left[\rho\, {y_m}^{\frac{z}{2}+N-3}(\frac{z}{2}+N-3) - \left((\frac{z}{2}+N-4)y_m - \frac{z}{2} - N + 5\right) y_m a r_+^{\frac{z}{2}+N-3}\right] B'(1)}{a r_+^{\frac{z}{2}+N-3} y_m(1-y_m)}}\,. \tag{5.23}$$

The above relations allude to the phase transition, namely, given ρ, μ has a maximum value when we assume the non-trivial solution $b \neq 0$. Now we can reveal the phase transition in our simple system. In order to evaluate the expectation value of the operator $< \mathcal{O}_+ >$, we eliminate the $a^2 b$ term from (5.20) and (5.21) and obtain

$$< \mathcal{O}_+ > \;=\; -\frac{{r_+}^{\Delta} b(m^2 {r_+}^2 (y_m - 1) + 2) {y_m}^{1-\Delta}}{((\Delta - 2) y_m - \Delta)} \,. \tag{5.24}$$

For non-vanishing b, we can compute $< \mathcal{O}_+ >$ from Eqs. (5.20)-(5.21) and one gets

$$a = |B'(1)| \sqrt{\frac{2\Delta^2 - m^2 r_+^2 \Big(y_m + 2\Delta(1 - y_m) \Big)}{r_+^{2-z} \Big[\Delta(1 - y_m)^2 + 2 y_m (1 - y_m) \Big]}}. \tag{5.25}$$

By plugging this result in Eq. (5.23) we derive

$$b = \frac{1}{2r_+} \sqrt{\frac{\sqrt{2} B'(1)}{(1 - y_m) \Sigma r_+^{\frac{z}{2}+N-3}} \Big[(\frac{z}{2} + N - 3) \rho y_m^{\frac{z}{2}+N-4} + \sqrt{2} r_+^{\frac{z}{2}+N-3} \Sigma \Big((\frac{z}{2} + N - 4)(1 - y_m) - 1 \Big) \Big]} \,, \tag{5.26}$$

$$\Sigma = |B'(1)| \sqrt{\frac{(m^2 {r_+}^2 (\Delta - 1) y_m - \Delta \left(-1 + m^2 {r_+}^2\right))}{{r_+}^{2-z} (y_m - 1)((\Delta - 2) y_m - \Delta)}}. \tag{5.27}$$

By using the density (5.14), one has that $\langle \mathcal{O}_+ \rangle$ can be expressed as

$$\langle \mathcal{O}_+ \rangle \;=\; -\frac{{y_m}^{1-\Delta} ({r_+}^2 (-1 + y_m) m^2 + 2\,) {r_+}^{\Delta}}{((\Delta - 2) y_m - \Delta) r_+} \sqrt{\Gamma}, \tag{5.28}$$

where the new function Γ is defined as

$$\Gamma = \frac{\Big(-\sqrt{2}((\frac{z}{2} + N - 4) y_m - \frac{z}{2} - N + 5) \Sigma\, {r_+}^{\frac{z}{2}+N-3} + (\frac{z}{2} + N - 3) {y_m}^{\frac{z}{2}+N-4} \rho \Big) \sqrt{2} B'(1)}{\Sigma\, {r_+}^{\frac{z}{2}+N-3} (1 - y_m)} \tag{5.29}$$

Now we can write Γ in the following equivalent form

$$\Gamma \;=\; A \frac{T_c - T_H}{\sqrt{T_H}}, \tag{5.30}$$

where T_H is the Kodama temperature (3.15) and A is a function of $\{r_+, \Delta, y_m\}$. The critical temperature T_c is defined as

$$\frac{T_c}{\sqrt{\rho}} = \frac{(\frac{z}{2}+N-3){y_m}^{\frac{z}{2}+N-3}}{4\sqrt{2}y_m\pi r_+^{\frac{z}{2}+N-2}(-(\frac{z}{2}+N-4)y_m+\frac{z}{2}+N-5)} \times \sqrt{\frac{{r_+}^{2-z}(1-y_m)\Big((\Delta-2)y_m-\Delta\Big)}{-m^2{r_+}^2(\Delta-1)y_m-\Delta\left(1-m^2{r_+}^2\right)}} . \tag{5.31}$$

Note that if we choose $y_m = 1/2$, $z = 0$ and $N = 4$, the critical temperature behaves like $T_c \sim \sqrt{\rho}$, according to the literature about s-wave holographic superconductors. This behavior will change only in the presence of the higher order corrected backgrounds like Weyl's models for s-wave. For example for Gauss-Bonnet and Weyl corrections it reads as $T_c = \sqrt[3]{\rho}$ [44].

Finally, using Eq. (5.31) we have

$$\langle \mathcal{O}_+ \rangle \sim \frac{\sqrt{T_c - T_H}}{\sqrt[4]{T_H}} . \tag{5.32}$$

The last expression has been written near the critical point T_c.

We see that $\langle \mathcal{O}_+ \rangle$ is zero at $T_H = T_c$, which is the critical point, and condensation occurs at $T_H < T_c$. The behaviour of $\langle \mathcal{O}_+ \rangle \propto (1-T_H/T_c)^{1/2}$ is also recovered, in agreement with the literature about the argument. The system below this critical temperature becomes superconductor, but the critical exponent of the model remains the same of the usual holographic superconductors without the higher order gravitational corrections.

6 Discussions

In this paper we have considered a holographic model for a non-relativistic system showing superconductivity. We have used a black hole background which comes from a scalar field model non minimally coupled with the Abelian gauge field in the presence of cosmological constant Λ in N dimensions, and we have studied analytically holographic superconductors in this new kind of asymptotic AdS solutions. We have considered static, (pseudo-)spherically symmetric (SSS) solutions with various topologies in two different cases, $\Lambda = 0$ and $\Lambda \neq 0$. We have obtained the quasi-local generalized Misner-Sharp mass as a Killing conserved charge. This quasi-local energy at the horizon $r = r_+$ is identified with black hole energy. Then we have derived the Wlad entropy, Killing-Hawking temperature and Kodama-Hayward temperature of black hole solutions. These temperatures are in principle different. We have shown that for our non-vacuum solution, the first law of black hole thermodynamics holds true by making use of Kodama temperature. After that we have studied the hairy black hole solutions in which near the horizon the Abelian gauge field breaks the symmetry. In the holographic picture, this symmetry breaking is equivalent to a second

order phase transition near the horizon. We also have analytically solved the system in the probe limits, near horizon and asymptotic region. We have found that there is also a critical temperature which is a function of the Lifshitz parameter z and under such temperature a condensation field appears.

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